Cooper pairs

Let us consider Fermi gas (noninteracting) characterized by Fermi energy denoted by ε_F.

Let us distinguish a pair of fermions located in the vicinity of Fermi surface.

We assume that the pair of fermions interacts through an attractive interaction.

For simplicity we disregard interaction of the pair with the rest of the system.

The presence of other fermions is felt by the pair through Pauli principle.

The wave function of the pair will be denoted by $\psi(\vec{r}_1\sigma_1, \vec{r}_2\sigma_2)$, where σ_1, σ_2 are spin projections on z-axis.

Since the interaction is assummed to be spin-independent therefore spins are decoupled from the spatial part of the wave function.

Since our fermion possess $s_i = \frac{\hbar}{2}$ spin therefore as a whole the pair can possess either the total spin zero:

$$|00\rangle = \frac{1}{\sqrt{2}}\Big(|\tfrac{1}{2} \underset{\sigma_1}{\underset{\uparrow}{+\tfrac{1}{2}}}\rangle|\tfrac{1}{2} \underset{\sigma_2}{\underset{\uparrow}{-\tfrac{1}{2}}}\rangle - |\tfrac{1}{2} -\tfrac{1}{2}\rangle|\tfrac{1}{2} +\tfrac{1}{2}\rangle\Big)$$

or the total spin $1\hbar$:

$$|1\,1\rangle = |\tfrac{1}{2} +\tfrac{1}{2}\rangle|\tfrac{1}{2} -\tfrac{1}{2}\rangle$$

$$|1\,0\rangle = \frac{1}{\sqrt{2}}\Big(|\tfrac{1}{2} +\tfrac{1}{2}\rangle|\tfrac{1}{2} -\tfrac{1}{2}\rangle + |\tfrac{1}{2} -\tfrac{1}{2}\rangle|\tfrac{1}{2} +\tfrac{1}{2}\rangle\Big)$$

$$|1\,-1\rangle = |\tfrac{1}{2} -\tfrac{1}{2}\rangle|\tfrac{1}{2} -\tfrac{1}{2}\rangle$$

Thus we can either have:

$$\psi_0(\vec{r}_1\sigma_1, \vec{r}_2\sigma_2) = \varphi_0(\vec{r}_1, \vec{r}_2)\, |00\rangle \quad \leftarrow \text{spin singlet state}$$

$$\psi_1(\vec{r}_1\sigma_1, \vec{r}_2\sigma_2) = \varphi_1(\vec{r}_1, \vec{r}_2)\, |1\sigma\rangle \quad \leftarrow \text{spin triplet state}$$

$$\sigma = \sigma_1 + \sigma_2 \quad , \quad \sigma = 0, \pm 1$$

Since the wave function has to be antisymmetric:

$$\psi(\vec{r}_1\sigma_1, \vec{r}_2\sigma_2) = -\psi(\vec{r}_2\sigma_2, \vec{r}_1\sigma_1)$$

therefore:

$$\varphi_0(\vec{r}_1, \vec{r}_2) = \varphi_0(\vec{r}_2, \vec{r}_1)$$

$$\varphi_1(\vec{r}_1, \vec{r}_2) = -\varphi_1(\vec{r}_2, \vec{r}_1)$$

This is a consequence of symmetry of the spin part:

$|00\rangle$ is antisymmetric

$|1\sigma\rangle$ is symmetric

Since the interaction is attractive we expect that its effect will be pronounced the most for φ_0.

Expressing $\varphi_0(\vec{r}_1, \vec{r}_2)$ in plane wave basis one gets:

$$\varphi_0(\vec{r}_1, \vec{r}_2) = \sum_{\substack{\vec{k}_1, \vec{k}_2 \\ |\vec{k}_1|, |\vec{k}_2| \geq k_F}} c_{\vec{k}_1\vec{k}_2} \left(e^{i\vec{k}_1\cdot\vec{r}_1} e^{i\vec{k}_2\cdot\vec{r}_2} + e^{i\vec{k}_1\cdot\vec{r}_2} e^{i\vec{k}_2\cdot\vec{r}_1} \right)$$

Introducing center of mass coordinates: $\begin{cases} \vec{R} = \frac{1}{2}(\vec{r}_1 + \vec{r}_2) & ; \quad \vec{K} = \vec{k}_1 + \vec{k}_2 \\ \vec{r} = \vec{r}_1 - \vec{r}_2 & ; \quad \vec{k} = \frac{1}{2}(\vec{k}_1 - \vec{k}_2) \end{cases}$

$$\varphi_0(\vec{R}, \vec{r}) = \sum_{\substack{\vec{K}, \vec{k} \\ |\vec{k}_1|, |\vec{k}_2| \geq k_F}} c_{\vec{K}\vec{k}} \Big[e^{i(\frac{1}{2}\vec{K} + \vec{k})\cdot(\vec{R} + \frac{1}{2}\vec{r})} e^{i(\frac{1}{2}\vec{K} - \vec{k})\cdot(\vec{R} - \frac{1}{2}\vec{r})} + $$

$$+ e^{i(\frac{1}{2}\vec{K} + \vec{k})(\vec{R} - \frac{1}{2}\vec{r})} e^{i(\frac{1}{2}\vec{K} - \vec{k})\cdot(\vec{R} + \frac{1}{2}\vec{r})} \Big]$$

$$\varphi_0(\vec{R},\vec{r}) = \sum_{\substack{\vec{K},\vec{k},\\ |\vec{k}_1|,|\vec{k}_2| \geq k_F}} c_{\vec{K}\vec{k}} \left[e^{i\vec{K}\cdot\vec{R}} e^{i\vec{k}\cdot\vec{r}} + e^{i\vec{K}\cdot\vec{R}} e^{-i\vec{k}\cdot\vec{r}} \right]$$

Hence we can write :

$$\varphi_0(\vec{R},\vec{r}) = \sum_{\substack{\vec{K},\vec{k}\\ |\vec{k}_1|,|\vec{k}_2| \geq k_F}} c_{\vec{K}\vec{k}}\, e^{i\vec{K}\cdot\vec{R}} e^{i\vec{k}\cdot\vec{r}} = \Big\{ \text{with condition: } c_{\vec{K}\vec{k}} = c_{\vec{K}-\vec{k}}$$

$$= \sum_{\substack{\vec{K},\vec{k}\\ |\vec{k}_1|,|\vec{k}_2| \geq k_F}} 2\, c_{\vec{K}\vec{k}}\, e^{i\vec{K}\cdot\vec{R}} \cos(\vec{k}\cdot\vec{r})$$

We are going to consider the case of $\vec{K}=0$ (pair at rest).

It implies that $\vec{k}_1 = -\vec{k}_2 = \vec{k}$

Hence we get :

$$\varphi_0(\vec{r}) = \sum_{\substack{\vec{k}\\ |\vec{k}| \geq k_F}} c_{\vec{k}}\, e^{i\vec{k}\cdot\vec{r}} \qquad \text{and} \qquad c_{\vec{k}} = c_{0\vec{k}} = c_{-\vec{k}}$$

We look for the solution of Schrödinger eq. (time independent):

$$(**) \quad \hat{H}\varphi_0(\vec{r}_1,\vec{r}_2) = E\,\varphi_0(\vec{r}_1,\vec{r}_2)$$

where $\hat{H} = -\frac{\hbar^2}{2m}\nabla_1^2 - \frac{\hbar^2}{2m}\nabla_2^2 + \hat{V} = \hat{H}_0 + \hat{V}$

describes attractive interaction

Hamiltonian in center of mass coordinates reads :

$$\hat{H} = -\frac{\hbar^2}{2M}\nabla_R^2 - \frac{\hbar^2}{2\mu}\nabla_r^2 + V(\vec{r}) \quad ; \quad M = 2m \,, \quad \mu = \frac{m}{2}$$

Hence instead of (**) we have

$$\hat{H}\varphi_0(\vec{R},\vec{r}) = E\,\varphi_0(\vec{R},\vec{r})$$

Note that since we put $\vec{K}=0$ then φ_0 does not depend on $\vec{R}$

Moreover we need to take into account somehow the presence of other particles. Therefore we assume that the only available (unoccupied) states are those with $k \geq k_F$.

Note that if $\hat{V}=0$ and we place the pair at Fermi surface ie. $|\vec{k}| = k_F$ then $E = 2\varepsilon_F = 2\frac{\hbar^2 k_F^2}{2m}$. It means that in the expression (*) the summation is limited to $|\vec{k}| = k_F$.

Substituting (*) to eq. (**) one gets:

$$\left[\hat{H}_0 \sum_{\substack{\vec{k} \\ k \geq k_F}} c_{\vec{k}} e^{i\vec{k}\cdot\vec{r}} + \hat{V} \sum_{\substack{\vec{k} \\ k \geq k_F}} c_{\vec{k}} e^{i\vec{k}\cdot\vec{r}} \right] = E \sum_{\substack{\vec{k} \\ k \geq k_F}} c_{\vec{k}} e^{i\vec{k}\cdot\vec{r}}$$

$$\sum_{\substack{\vec{k} \\ k \geq k_F}} \left(E - 2\frac{\hbar^2 k^2}{2m} \right) c_{\vec{k}} e^{i\vec{k}\cdot\vec{r}} = \sum_{\substack{\vec{k} \\ k \geq k_F}} c_{\vec{k}} V(\vec{r}) e^{i\vec{k}\cdot\vec{r}}$$

Multiplying the eq. by $\frac{1}{V} e^{-i\vec{k}'\cdot\vec{r}}$ (V is volume of the system) and integrating over $\vec{r}$ (note that $\int e^{-i\vec{k}'\cdot\vec{r}} e^{i\vec{k}\cdot\vec{r}} d^3r = V \delta_{\vec{k}\vec{k}'}$) one gets:

$$\left(E - 2\frac{\hbar^2 k'^2}{2m} \right) c_{\vec{k}'} = \sum_{\substack{\vec{k} \\ k \geq k_F}} c_{\vec{k}} \underbrace{\frac{1}{V} \int e^{-i\vec{k}'\cdot\vec{r}} V(\vec{r}) e^{i\vec{k}\cdot\vec{r}} d^3r}_{V_{\vec{k}'\vec{k}}}$$

Thus finally:

$$(***) \quad \boxed{c_{\vec{k}} = \frac{1}{E - 2\frac{\hbar^2 k^2}{2m}} \sum_{\substack{\vec{k}' \\ k' \geq k_F}} V_{\vec{k}\vec{k}'} c_{\vec{k}'} \; ; \quad |\vec{k}| \geq k_F}$$

In order to solve (***) one need to substitute the form of the interaction in order to evaluate matrix element $V_{\vec{k}'\vec{k}}$.
However the main feature of the solution can be obtained using a simplified form of $V_{\vec{k}'\vec{k}}$:

$$V_{\vec{k}'\vec{k}} = \begin{cases} -V_0 & ; \quad k', k \leq k_c \\ 0 \end{cases}$$

where k_c is a parameter.

Moreover let us replace summation on rhs by integration:

$$\sum_{\substack{\vec{k} \\ k \geq k_F}} \rightarrow \frac{V}{(2\pi)^3} \int_{k \geq k_F} d^3 k$$

Hence one gets:

$$c_k = \frac{-V_0}{E - 2\frac{\hbar^2 k^2}{2m}} \frac{V}{(2\pi)^3} 4\pi \int_{k_F}^{k_c} k'^2 dk' \, c_{k'}$$

where we assumed that $c_{\vec{k}} = c_k$ (no dependence on spatial orientation of $\vec{k}$) and changing variables: $k \to \varepsilon = \frac{\hbar^2 k^2}{2m} \Rightarrow dk = \sqrt{\frac{2m}{\hbar^2}} \frac{d\varepsilon}{2\sqrt{\varepsilon}}$

$$c(\varepsilon) = -\frac{V_0}{E - 2\varepsilon} \frac{V}{2\pi^2} \int_{\varepsilon_F}^{\varepsilon_c} \frac{2m\varepsilon'}{\hbar^2} \sqrt{\frac{2m}{\hbar^2}} \frac{d\varepsilon'}{2\sqrt{\varepsilon'}} c(\varepsilon')$$

$$c(\varepsilon) = \frac{-V_0}{E - 2\varepsilon} \frac{V}{(2\pi)^2} \int_{\varepsilon_F}^{\varepsilon_c} \left(\frac{2m}{\hbar^2}\right)^{3/2} \sqrt{\varepsilon'} \, c(\varepsilon') d\varepsilon' = \frac{V_0}{2\varepsilon - E} \int_{\varepsilon_F}^{\varepsilon_c} g(\varepsilon') c(\varepsilon') d\varepsilon'$$

$$\int_{\varepsilon_F}^{\varepsilon_c} g(\varepsilon) c(\varepsilon) \, d\varepsilon = V_0 \int_{\varepsilon_F}^{\varepsilon_c} \frac{1}{2\varepsilon - E} g(\varepsilon) \, d\varepsilon \int_{\varepsilon_F}^{\varepsilon_c} g(\varepsilon') c(\varepsilon') d\varepsilon'$$

$$1 = V_0 \int_{\varepsilon_F}^{\varepsilon_c} \frac{g(\varepsilon)}{2\varepsilon - E} d\varepsilon \approx \frac{1}{2} V_0 \, g(\varepsilon_F) \log\left[\frac{2\varepsilon_c - E}{2\varepsilon_F - E}\right]$$

where the density of states per spin for a Fermi gas reads:

$$\boxed{g(\varepsilon) = \frac{V}{(2\pi)^2} \left(\frac{2m}{\hbar^2}\right)^{3/2} \sqrt{\varepsilon}}$$

Hence we get:

$$1 \approx \frac{1}{2} V_0 \, g(\varepsilon_F) \log \frac{2\varepsilon_c - E}{2\varepsilon_F - E}$$

Let us denote: $\Delta = 2\varepsilon_F - E$ and $\hbar\omega_D = \varepsilon_c - \varepsilon_F$

Then

$$1 \approx \frac{1}{2} V_0 \, g(\varepsilon_F) \log \frac{\hbar\omega_D + \Delta}{\Delta}$$

$$\frac{2\hbar\omega_D + \Delta}{\Delta} \approx \exp\left[\frac{1}{\frac{1}{2} V_0 \, g(\varepsilon_F)}\right]$$

$$\Delta \left(\exp\left[\frac{1}{\frac{1}{2} V_0 \, g(\varepsilon_F)}\right] - 1 \right) \approx 2\hbar\omega_D$$

$$\Delta \approx 2\hbar\omega_D \frac{1}{\exp\left[\frac{1}{\frac{1}{2} V_0 \, g(\varepsilon_F)} - 1\right]}$$

In the limit of weak interaction $\frac{1}{2} V_0 \, g(\varepsilon_F) \ll 1$ it leads to:

$$\boxed{\Delta \approx 2\hbar\omega_D \, e^{-\frac{1}{\frac{1}{2} V_0 \, g(\varepsilon_F)}}}$$

The above relation indicates that for arbitrarily weak interaction the energy of the pair of particles placed at the Fermi surface is lowered by $\Delta = 2\varepsilon_F - E > 0$.

Moreover it is crucial that the pair is immersed in Fermi gas $(\varepsilon_F > 0)$.

Otherwise $\varepsilon_F = 0 \Rightarrow g(\varepsilon_F) = 0 \Rightarrow \Delta = 0$.

Structure of the pair wave function

The spatial part of the wave function reads (see (*)):

$$\varphi_0(\vec{r}) = \frac{V}{(2\pi)^3} \int_{|\vec{k}| \geq k_F} d^3k \; c_{\vec{k}} \, e^{i\vec{k}\cdot\vec{r}} \; ; \qquad c_{\vec{k}} = \frac{-V_0}{E - 2\frac{\hbar^2 k^2}{2m}} \frac{V}{(2\pi)^3} \int_{\substack{|\vec{k}'| \geq k_F \\ |\vec{k}'| < k_c}} d^3k' \; c_{\vec{k}'}$$

From the equation for coefficients $c_{\vec{k}}$ it is clear that they do not depend on orientation of $\vec{k}$:

$$c_{\vec{k}} = c_k = -\frac{V_0}{E - 2\frac{\hbar^2 k^2}{2m}}\alpha \quad ; \quad \alpha = \frac{V}{(2\pi)^3}\int_{\substack{|\vec{k}|\geq k_F \\ |\vec{k}|<k_c}} d^3k \, c_{\vec{k}}$$

Hence we have: $$\varphi_0(\vec{r}) = \frac{V}{(2\pi)^3}\int_{\substack{|\vec{k}|\geq k_F \\ |\vec{k}|<k_c}} d^3k \, c_k \, e^{i\vec{k}\cdot\vec{r}}$$

$$\varphi_0(\vec{r}) = \frac{V}{(2\pi)^3}\int_{\substack{|\vec{k}|\geq k_F \\ |\vec{k}|<k_c}} d^3k \, \frac{V_0\alpha}{2\frac{\hbar^2k^2}{2m} - E} e^{i\vec{k}\cdot\vec{r}} = \frac{V}{(2\pi)^3} V_0\alpha \int_{\substack{|\vec{k}|\geq k_F \\ |\vec{k}|<k_c}} d^3k \, \frac{e^{i\vec{k}\cdot\vec{r}}}{2\frac{\hbar^2k^2}{2m} - 2\varepsilon_F + \Delta}$$

$$\varphi_0(\vec{r}) = \frac{V}{(2\pi)^3} V_0\alpha \, 2\pi \int_{k_F}^{k_c} k^2 dk \frac{2}{\left(2\frac{\hbar^2k^2}{2m} - 2\varepsilon_F + \Delta\right)} \frac{\sin(kr)}{kr} =$$

$$= \underbrace{\frac{V}{2\pi^2} V_0\alpha}_{\beta} \frac{1}{r}\int_{k_F}^{k_c} dk \frac{k\sin(kr)}{2\frac{\hbar^2k^2}{2m} - 2\varepsilon_F + \Delta} = \frac{\beta}{r}\frac{1}{2i}\int_{k_F}^{k_c} \frac{\left(e^{ikr} - e^{-ikr}\right)k\,dk}{2\frac{\hbar^2k^2}{2m} - 2\varepsilon_F + \Delta} =$$

$$= \frac{\beta}{2ir}\left[e^{ik_Fr}\int_{k_F}^{k_c} \frac{k e^{i\delta kr}}{2\frac{\hbar^2k^2}{2m} - 2\varepsilon_F + \Delta} dk - e^{-ik_Fr}\int_{k_F}^{k_c} \frac{k e^{-i\delta kr} \, dk}{2\frac{\hbar^2k^2}{2m} - 2\varepsilon_F + \Delta}\right]$$

where $\delta k = k - k_F$

If $\delta k r \ll 1$: $e^{\pm i\delta kr} \approx 1$ and $\varphi_0(\vec{r}) \approx \text{const.} \frac{\sin(k_F r)}{r}$ – free particle solution

If $\delta k r \gg 1$: $$\int_{k_F}^{k_c} f(k) e^{\pm i\delta kr} dk = \frac{1}{(\pm ir)} e^{\pm i\delta kr} f(k)\Big|_{k_F}^{k_c} - \frac{1}{(\pm ir)}\int_{k_F}^{k_c} f'(k) e^{\pm i\delta kr} dk$$

Therefore in this limit $\int_{k_F}^{k_c} f(k) e^{\pm i\delta kr} dk$ vanishes at least as $\frac{1}{r}$

Consequently the distances at which the behavior of $\varphi_0(\vec{r})$ changes from $\sim \frac{\sin(k_F r)}{r}$ to $\sim \frac{1}{r^2}$ read $\delta k r \approx 1$

Note however that the integrand vanishes effectively for $2\frac{\hbar^2k^2}{2m} - 2\varepsilon_F > \Delta$

Hence we can set the effective limit on k:

$$\frac{\hbar^2 k^2}{m} = 2\varepsilon_F + \Delta$$

$$k = \sqrt{2\varepsilon_F \frac{m}{\hbar^2} + \frac{m}{\hbar^2}\Delta}$$

Hence from the condition: $(k - k_F) r = 1$

$$r = \frac{1}{k - k_F}$$

$$r = \frac{k + k_F}{k^2 - k_F^2}$$

$$r \lesssim \frac{k_F}{2\varepsilon_F \frac{m}{\hbar^2} + \frac{m}{\hbar^2}\Delta - k_F^2} = \frac{\hbar^2}{m} \frac{k_F}{\Delta}$$

$$r \lesssim 2 \frac{1}{k_F} \frac{\varepsilon_F}{\Delta}$$

Hence we can call the distance: $\boxed{R = \frac{1}{k_F} \frac{\varepsilon_F}{\Delta}}$ - the size of Cooper pair

Appendix

1° In various books $\langle r^2\rangle$ is evaluated in the following way

$$\langle r^2\rangle = \frac{\int r^2 |\varphi_0(\vec r)|^2 d^3r}{\int |\varphi_0(\vec r)|^2 d^3r} \quad ; \quad \varphi_0(\vec r) = \int d^3k\, f(k)\, e^{i\vec k\cdot\vec r}$$

$$f(k) = \frac{\alpha}{\frac{\hbar^2k^2}{m} - E} \quad \text{for } k\in(k_F, k_c)$$

otherwise $f(k)=0$.

$$\int r^2|\varphi_0(\vec r)|^2 d^3r = \int d^3r \int d^3k\, d^3k'\, f(k) f(k')\, \vec r e^{-i\vec k'\cdot\vec r} \cdot \vec r e^{i\vec k\cdot\vec r} =$$

$$= \int d^3r \int d^3k\, d^3k'\, f(k) f(k')\, \vec\nabla_{k'} e^{-i\vec k'\cdot\vec r} \cdot \vec\nabla_k e^{i\vec k\cdot\vec r} =$$

$$= \int d^3k\, d^3k'\, \vec\nabla_k f(k)\cdot\vec\nabla_{k'} f(\vec k')\,(2\pi)^3\delta(k-k') =$$

$$= \int d^3k \left[\vec\nabla_k f(k)\right]^2 (2\pi)^3 = (2\pi)^3 4\pi \int_0^\infty k^2 dk \left(\frac{df}{dk}\right)^2 (\vec\nabla k)^2 = (2\pi)^3 4\pi \int_0^\infty k^2 \left(\frac{df}{dk}\right)^2 dk$$

$$\varepsilon = \frac{\hbar^2k^2}{2m} \Rightarrow k = \sqrt{\frac{2m\varepsilon}{\hbar^2}}$$

$$dk = \sqrt{\frac{2m}{\hbar^2}}\frac{1}{2\sqrt\varepsilon}d\varepsilon \quad ; \quad \frac{df}{dk} = \frac{df}{d\varepsilon}\frac{d\varepsilon}{dk} = \frac{\hbar^2}{m}k\frac{df}{d\varepsilon} = \frac{\hbar^2}{m}\sqrt{\frac{2m\varepsilon}{\hbar^2}}\frac{df}{d\varepsilon} = \sqrt{\frac{2\hbar^2\varepsilon}{m}}\frac{df}{d\varepsilon}$$

$$\int r^2|\varphi_0(\vec r)|^2 d^3r = (2\pi)^3 4\pi \int_0^\infty \frac{2m\varepsilon}{\hbar^2}\frac{2\hbar^2\varepsilon}{m}\left(\frac{df}{d\varepsilon}\right)^2 \sqrt{\frac{2m}{\hbar^2}}\frac{1}{2\sqrt\varepsilon}d\varepsilon \overset{?}{\approx} (2\pi)^3 8\pi\sqrt{\frac{2m}{\hbar^2}}\,\varepsilon_F^{3/2}\int_0^\infty\left(\frac{df}{d\varepsilon}\right)^2 d\varepsilon$$

see (*)

$$\int |\varphi_0(\vec r)|^2 d^3r = \int d^3k\, |f(k)|^2(2\pi)^3 = (2\pi)^3 4\pi\int_0^\infty k^2|f(k)|^2 dk \approx (2\pi)^3 4\pi\left(\frac{2m}{\hbar^2}\right)^{3/2}\frac{1}{2}\sqrt{\varepsilon_F}\int_0^\infty f^2 d\varepsilon$$

Hence

$$\langle r^2\rangle = 4\frac{\hbar^2}{2m}\varepsilon_F \frac{\int_{\varepsilon_F}^\infty \frac{4}{(2\varepsilon-E)^4}d\varepsilon}{\int_{\varepsilon_F}^\infty \frac{1}{(2\varepsilon-E)^2}d\varepsilon} = 16\frac{\hbar^2}{2m}\varepsilon_F\frac{\frac{1}{(2\varepsilon_F-E)^3}\frac{1}{6}}{\frac{1}{2\varepsilon_F-E}\frac{1}{2}} = \frac{16}{3}\frac{\hbar^2\varepsilon_F}{m}\frac{1}{\Delta^2}$$

$$\frac{df}{d\varepsilon} = \frac{-2}{(2\varepsilon-E)^2}$$

$$\sqrt{\langle r^2\rangle} \approx \frac{4}{\sqrt3}\sqrt{\frac{\hbar^2}{m}}\frac{\sqrt{\varepsilon_F}}{\Delta} = \frac{4}{\sqrt3}\sqrt{\frac{\hbar^2}{m}}\frac{\varepsilon_F}{\Delta}\frac{1}{\sqrt{\frac{\hbar^2k_F^2}{2m}}} = 4\sqrt{\frac{2}{3}}\frac{1}{k_F}\frac{\varepsilon_F}{\Delta}$$

There is however a subtlety which is usually ignored. Namely f is discontinuous (at least at ε_F) which makes $\frac{df}{d\varepsilon}$ singular. (*)